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Uniform Representation of the Gravitational Potential and its Derivatives

SAMUEL PINES*
Analytical Mechanics Associates Inc., Jericho, N.Y.

The expressions for the gravitational potential and its derivatives of a nonspherical body in terms of spherical coordinates prove cumbersome at or near the zonal poles. A coordinate system is presented herein which removes this difficulty yet retains the numerical values and functional forms of the mass coefficients ($C_{n,m}$, $S_{n,m}$) as well as the orthogonality and recursion properties similar to the spherical harmonic representation.

Introduction

THE coordinates are used to express the potential and its first and second partial derivatives with respect to the Cartesian coordinates of a point mass. Recursion equations are derived for the new functions replacing the associated Legendre polynomials of the declination, or colatitude, and the trigonometric functions of the longitude. Moreover, the recursion equations are so chosen as to be stable for large values of the integer argument. Reference 1 contains a review of several methods for generating the first and second partials of a nonspherical body potential in a recursive formulation. The main advantage of the method outlined herein lies in a simplification of the formulation, which results in requiring only thirteen summation functions in place of the fourteen outlined in Ref. 1.

In order to facilitate the application of the new representation to statistical estimation theory for the determination of the Cartesian state and the mass coefficients, the necessary matrices and the variational differential equations are derived for a massless orbiter about a nonspherical, rotating body.

The attention of the author has been brought to a paper by Vinti,[2] in which the singularity of polar orbits is removed by a transformation in the complex domain and applied to the closed form solution of a satellite in orbit about an oblate gravitational body.

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* President. Member AIAA.

The Conventional Representation

Let the reference frame be an orthogonal Cartesian three-dimensional coordinate system fixed in the nonspherical body. In the most general case, the origin need not be at the center of mass nor need the plane normal to the pole be coincident with the body equator. In this case, the mass coefficients, J_1, C_{11}, S_{11} are not necessarily equal to zero and are to be included in the representation. Furthermore, to obtain a more condensed representation, let

$$\begin{aligned} C_{00} &= 1 \\ C_{n0} &= -J_n \end{aligned} \tag{1}$$

The Cartesian position vector, $R(x, y, z)$ is given in terms of the spherical coordinates, r, α, λ, as follows:

$$R = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = r \begin{Bmatrix} \cos\alpha\cos\lambda \\ \cos\alpha\sin\lambda \\ \sin\alpha \end{Bmatrix} \tag{2}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

The conventional representation of the gravitational potential φ is given by

$$\varphi = \frac{\mu}{r}\left\{1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=1}^{n} P_{n,m}(\sin\alpha)(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\right\} \tag{3}$$

where

$$\begin{aligned} a &= \text{body radius} \\ \mu &= \text{gravitational body constant} \end{aligned} \tag{4}$$

$$P_{n,m}(\sin\alpha) = (1-\sin^2\alpha)^{m/2}\frac{1}{2^n n!}\frac{d^{n+m}}{d\sin\alpha^{n+m}}(\sin^2\alpha - 1)^n$$

The acceleration force vector in the Cartesian coordinate frame of reference fixed in the body is given by the partials of φ with respect to the three components of R, shown in vector notation as

$$F = \frac{\partial\varphi}{\partial R} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial\varphi}{\partial\sin\alpha}\frac{\partial\sin\alpha}{\partial R} + \frac{\partial\varphi}{\partial\lambda}\frac{\partial\lambda}{\partial R} \tag{5}$$

where the vectors, $\partial r/\partial R$, $\partial\sin\alpha/\partial R$, and $\partial\lambda/\partial R$ are defined as

$$\begin{aligned} \partial r/\partial R &= (1/r)R = \hat{R} \\ \partial\sin\alpha/\partial R &= (1/r)\hat{k} - (\sin\alpha/r)\hat{R} \\ \partial\lambda/\partial R &= (1/r\cos^2\alpha)\hat{k}\times\hat{R} \\ \hat{k} &= \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \end{aligned} \tag{6}$$

It is apparent by inspection that, for $|\alpha| \cong \pi/2$, the vectors are poorly defined. Moreover, the partials of the associated Legendre polynomials for $m = 1$ possess similar nonuniform properties. A change of coordinates is carried out which avoids this difficulty yet retains similar recursive and orthogonality properties.

A Uniform Representation

Let the variables be r, the scalar magnitude of the vector R, and the three components of the unit vector $\hat{R}$. Then

$$\begin{aligned} r &= (x^2+y^2+z^2)^{1/2} \\ s &= x/r \\ t &= y/r \\ u &= z/r \end{aligned} \tag{7}$$

where

$$R = r\begin{Bmatrix} s \\ t \\ u \end{Bmatrix}, \qquad s^2+t^2+u^2 = 1 \tag{7a}$$

In place of the associated Legendre polynomials, $P_{n,m}(u)$, let

$$A_{n,m}(u) = \frac{1}{2^n n!}\frac{d^{n+m}}{du^{n+m}}(u^2-1)^n \tag{8}$$

In place of the $\sin m\lambda$, $\cos m\lambda$, let

$$\begin{aligned} r_m(s,t) &= \text{Real part } (s+it)^m \\ i_m(s,t) &= \text{Imaginary part } (s+it)^m \end{aligned} \tag{9}$$

where

$$i = (-1)^{1/2}$$

Since

$$\begin{aligned} \cos m\lambda\cos m\alpha &= r_m(s,t) \\ \sin m\lambda\cos m\alpha &= i_m(s,t) \end{aligned} \tag{10}$$

the new representation of the potential becomes

$$\varphi = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=1}^{n} A_{n,m}(u)\big(C_{n,m}r_m(s,t) + S_{n,m}i_m(s,t)\big)\right] \tag{11}$$

It is plain that no singularities occur for the potential or the partial derivatives of φ with respect to r, s, t, or u except at the origin, $r = 0$. In any event, the original representation is not valid for $|r| < a$, so that little is lost.

The partials of the new variables with respect to the vector R are given by

$$\begin{aligned} \partial r/\partial R &= (1/r)\hat{R} \\ \partial s/\partial R &= (1/r)\hat{\imath} - (s/r)\hat{R} \\ \partial r/\partial R &= (1/r)\hat{\jmath} - (t/r)\hat{R} \\ \partial u/\partial R &= (1/r)\hat{k} - (u/r)\hat{R} \end{aligned} \tag{12}$$

where

$$\hat{\imath} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad\text{and}\quad \hat{\jmath} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{12a}$$

Combining the scalar terms as coefficients of the vectors $\hat{R}$, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, the acceleration force is

$$F = \left(\frac{\partial\varphi}{\partial r} - \frac{s}{r}\frac{\partial\varphi}{\partial s} - \frac{t}{r}\frac{\partial\varphi}{\partial t} - \frac{u}{r}\frac{\partial\varphi}{\partial u}\right)\hat{R} + \frac{1}{r}\frac{\partial\varphi}{\partial s}\hat{\imath} + \frac{1}{r}\frac{\partial\varphi}{\partial t}\hat{\jmath} + \frac{1}{r}\frac{\partial\varphi}{\partial u}\hat{k} \tag{13}$$

Before proceeding further, some recursion properties are derived.

Recursion Relationships

Consider the functions $A_{n,m}(u)$, which are to be called the derived Legendre polynomials since they are the mth derivatives of the Legendre polynomials $P_n(n)$.

$$\begin{aligned} A_{n,0}(u) &= P_n(u) = (1/2^n n!)(d^n/du^n)(u^2-1)^n \\ A_{n,m}(u) &= \frac{d^m}{du^m}P_n(u) = \frac{1}{2^n n!}\frac{d^{n+m}}{du^{n+m}}(u^2-1)^n \end{aligned} \tag{14}$$

The Legendre functions satisfy the known recursion equations

$$(d/du)P_{n+1}(u) = (n+1)P_n(u) + u(d/du)P_n(u) \tag{15}$$

The equivalent recursion equation for the $A_{n,m}(u)$ is given by

$$A_{n+1,1} = (n+1)A_{n,0} + uA_{n,1} \tag{16}$$

Successive differentiations of Eq. (16) with respect to u, yield

$$A_{n+1,m+1} = (n+1+m)A_{n,m} + uA_{n,m+1} \tag{17}$$

Equation (17) is useful for simplifying the gradients of φ but should not be used for recursion, since it is unstable for large n and m. It is obvious that any error in the computation of $A_{n,m}(u)$ will be multiplied by $n+m+1$ in forming $A_{n+1,m+1}(u)$ by Eq. (17). We will now derive a more stable formula.

Again, the Legendre polynomials satisfy the recursion equation

$$nP_n(u) = u(d/du)P_n(u) - (d/du)P_{n-1}(u) \tag{18}$$

For the $A_{n,m}(u)$, the equivalent equation is

$$nA_{n,0}(u) = uA_{n,1}(u) - A_{n-1,1}(u) \tag{19}$$

Successively differentiating Eq. (19) with respect to u results in

$$(n-m)A_{n,m}(u) = uA_{n,m+1}(u) - A_{n-1,m+1}(u) \tag{20}$$

or

$$A_{n,m}(u) = [1/(n-m)](uA_{n,m+1} - A_{n-1,m+1}) \tag{21}$$

Equation (21) is a stable recursion equation since any error in

$A_{n,m+1}$ or $A_{n+1,m+1}$ is divided by $n-m$, and thus decreased if $m < n+1$. To insure that $m < n+1$, the following process is recommended:

Since

$$(d^{2n+l}/du^{2n+l})(u^2-1)^n = 0 \tag{22}$$

for $l \geqq 1$, we have for all positive n

$$A_{n,m}(u) = 0 \qquad m > n \tag{22a}$$

Furthermore, we have for $n \geqq 1$

$$A_{n,n} = 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

and

$$A_{n,n-1} = uA_{n,n} \tag{23}$$

To obtain $A_{n,m}(u)$ for $0 < m < n-2$, we may now use Eq. (21) recursively for a fixed n and a decreasing m, $m = n-2, n-3, \ldots, 0$, with the two initial values given by Eq. (23).

The recursion relationships for $r_m(s,t)$ and $i_m(s,t)$ follow from complex variable algebra

$$\begin{aligned} r_m(s,t) &= sr_{m-1}(s,t) - ti_{m-1}(s,t) \\ i_m(s,t) &= si_{m-1}(s,t) + tr_{m-1}(s,t) \end{aligned} \tag{24}$$

Furthermore, the partials of r_m and i_m with respect to s and t are given by

$$\begin{aligned} \partial r_m/\partial s = \partial i_m/\partial t &= mr_{m-1}(s,t) \\ \partial i_m/\partial s = -\partial r_m/\partial t &= mi_{m-1}(s,t) \\ s(\partial r_m/\partial s) - t(\partial i_m/\partial t) &= r_m \\ s(\partial i_m/\partial s) + t(\partial r_m/\partial t) &= i_m \end{aligned} \tag{25}$$

For compactness of notation, we define the variable $\rho = a/r$. We form the simple recursion equations

$$\begin{aligned} \rho_0 &= \mu/r \\ \rho_1 &= \rho\rho_0 \\ \rho_n &= \rho\rho_{n-1} \quad \text{for} \quad n > 1 \end{aligned} \tag{26}$$

and

$$\begin{aligned} \rho_n/r &= (\rho_{n+1})/a \\ \partial\rho_n/\partial r &= -[(n+1)/a]\rho_{n+1} \end{aligned} \tag{26a}$$

Finally, we formulate certain convenient mass coefficient functions, as follows:

$$\begin{aligned} D_{n,m}(s,t) &= C_{n,m}r_m(s,t) + S_{n,m}i_m(s,t) \\ E_{n,m}(s,t) &= C_{n,m}r_{m-1}(s,t) + S_{n,m}i_{m-1}(s,t) \\ F_{n,m}(s,t) &= S_{n,m}r_{m-1}(s,t) - C_{n,m}i_{m-1}(s,t) \\ G_{n,m}(s,t) &= C_{n,m}r_{m-2}(s,t) + S_{n,m}i_{m-2}(s,t) \\ H_{n,m}(s,t) &= S_{n,m}r_{m-2}(s,t) - C_{n,m}i_{m-2}(s,t) \end{aligned} \tag{27}$$

From Eqs. (24) and (25), we have

$$\begin{aligned} \partial D_{n,m}/\partial s &= mE_{n,m} \\ \partial D_{n,m}/\partial t &= mF_{n,m} \end{aligned}$$

Since

$$\frac{\partial^2 D_{n,m}}{\partial s^2} + \frac{\partial^2 D_{n,m}}{\partial t^2} = 0, \quad \text{then} \quad \frac{\partial E_{n,m}}{\partial s} = (m-1)G_{n,m} = -\frac{\partial F_{n,m}}{\partial t} \tag{28}$$

and since

$$\frac{\partial^2 D_{n,m}}{\partial s\,\partial t} = \frac{\partial^2 D_{n,m}}{\partial t\,\partial s}, \quad \text{then} \quad \frac{\partial E_{n,m}}{\partial t} = (m-1)H_{n,m} = \frac{\partial F_{n,m}}{\partial s}$$

With the preceding, we may now proceed to obtain the required results.

The Derivatives of the Potential

The gravitational potential in our new representation is given by

$$\varphi = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{n,m}(u)D_{n,m}(s,t) \tag{29}$$

To obtain the acceleration force, we have for the coefficients of the three vectors of Eq. (13), $\hat{i}, \hat{j}$, and $\hat{k}$

$$\begin{aligned} a_1 &= \frac{1}{r}\frac{\partial\varphi}{\partial s} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m}(u)mE_{n,m} \\ a_2 &= \frac{1}{r}\frac{\partial\varphi}{\partial t} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m}(u)mF_{n,m} \\ a_3 &= \frac{1}{r}\frac{\partial\varphi}{\partial u} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m+1}(u)D_{n,m} \end{aligned} \tag{30}$$

To obtain the coefficients of $\hat{R}$, we have

$$\begin{aligned} a_4 &= \frac{\partial\varphi}{\partial r} - \frac{s}{r}\frac{\partial\varphi}{\partial s} - \frac{t}{r}\frac{\partial\varphi}{\partial t} - \frac{u}{r}\frac{\partial\varphi}{\partial u} \\ &= \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} [(n+1+m)A_{n,m}(u) + uA_{n,m+1}(u)]D_{n,m} \end{aligned} \tag{30a}$$

From the recursion Eq. (17), we have

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n+1,m+1}D_{n,m} \tag{30b}$$

Finally, the acceleration vector is given by

$$F = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + a_4\hat{R} \tag{31}$$

Before proceeding to the second partials of the potential, we require some results for the partial derivatives of the vectors in Eq. (31) with respect to the vector R. We note that since $\hat{i}, \hat{j}, \hat{k}$ are constant vectors

$$\partial\hat{i}/\partial R = \partial\hat{j}/\partial R = \partial\hat{k}/\partial R = 0 \tag{32}$$

For the $\partial\hat{R}/\partial R$, we have

$$\partial\hat{R}/\partial R = (1/r)(\hat{i}\hat{i}^T + \hat{j}\hat{j}^T + \hat{k}\hat{k}^T) - (1/r)\hat{R}\hat{R}^T \tag{33}$$

$$= (1/r)I(3 \times 3) - (1/r)\hat{R}\hat{R}^T \tag{33a}$$

The second partials of the potential φ with respect to the Cartesian position vector R are given by

$$\partial^2\varphi/\partial R^2 = (\partial/\partial R)(a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + a_4\hat{R}) \tag{34}$$

$$\begin{aligned} \frac{\partial F}{\partial R} = {} & \frac{1}{r}\left(\frac{\partial a_1}{\partial s} + a_4\right)\hat{i}\hat{i}^T + \frac{1}{r}\frac{\partial a_1}{\partial t}\hat{j}\hat{i}^T + \frac{1}{r}\frac{\partial a_1}{\partial u}\hat{k}\hat{i}^T + \\ & \left(\frac{\partial a_1}{\partial r} - \frac{s}{r}\frac{\partial a_1}{\partial s} - \frac{t}{r}\frac{\partial a_1}{\partial t} - \frac{u}{r}\frac{\partial a_1}{\partial u}\right)\hat{R}\hat{i}^T + \frac{1}{r}\frac{\partial a_2}{\partial s}\hat{i}\hat{j}^T + \\ & \frac{1}{r}\left(\frac{\partial a_2}{\partial t} + a_4\right)\hat{j}\hat{j}^T + \frac{1}{r}\frac{\partial a_2}{\partial u}\hat{k}\hat{j}^T + \left(\frac{\partial a_2}{\partial r} - \frac{s}{r}\frac{\partial a_2}{\partial s} - \frac{t}{r}\frac{\partial a_2}{\partial t} - \right. \\ & \left.\frac{u}{r}\frac{\partial a_2}{\partial u}\right)\hat{R}\hat{j}^T + \frac{1}{r}\frac{\partial a_3}{\partial s}\hat{i}\hat{k}^T + \frac{1}{r}\frac{\partial a_3}{\partial t}\hat{j}\hat{k}^T + \frac{1}{r}\left(\frac{\partial a_3}{\partial u} + a_4\right)\hat{k}\hat{k}^T + \\ & \left(\frac{\partial a_3}{\partial r} - \frac{s}{r}\frac{\partial a_3}{\partial s} - \frac{t}{r}\frac{\partial a_3}{\partial t} - \frac{u}{r}\frac{\partial a_3}{\partial u}\right)\hat{R}\hat{k}^T + \frac{1}{r}\frac{\partial a_4}{\partial s}\hat{i}\hat{R}^T + \\ & \frac{1}{r}\frac{\partial a_4}{\partial t}\hat{j}\hat{R}^T + \frac{1}{r}\frac{\partial a_4}{\partial u}\hat{k}\hat{R}^T + \left(\frac{\partial a_4}{\partial r} - \frac{s}{r}\frac{\partial a_4}{\partial s} - \frac{t}{r}\frac{\partial a_4}{\partial t} - \right. \\ & \left.\frac{u}{r}\frac{\partial a_4}{\partial u} - \frac{a_4}{r}\right)\hat{R}\hat{R}^T \end{aligned} \tag{35}$$

Although this seems hopelessly complicated, the results prove surprisingly simple in the new functions. Whereas sixteen summation coefficients are required, only nine need be computed due to the equality between mixed partial derivatives. We have

$$\begin{aligned} a_{11} &= \frac{1}{r}\frac{\partial a_1}{\partial s} = -\frac{1}{r}\frac{\partial a_2}{\partial t} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} m(m-1)A_{n,m}G_{n,m} \\ a_{12} &= \frac{1}{r}\frac{\partial a_1}{\partial t} = \frac{1}{r}\frac{\partial a_2}{\partial s} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} m(m-1)A_{n,m}H_{n,m} \\ a_{13} &= \frac{1}{r}\frac{\partial a_1}{\partial u} = \frac{1}{r}\frac{\partial a_3}{\partial s} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n,m+1}E_{n,m} \\ a_{14} &= \left(\frac{\partial a_1}{\partial r} - \frac{s}{r}\frac{\partial a_1}{\partial s} - \frac{t}{r}\frac{\partial a_1}{\partial t} - \frac{u}{r}\frac{\partial a_1}{\partial u}\right) = \frac{1}{r}\frac{\partial a_4}{\partial s} = \\ & \qquad -\sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n+1,m+1}E_{n,m} \end{aligned} \tag{36}$$

$$a_{23} = \frac{1}{r}\frac{\partial a_2}{\partial u} = \frac{1}{r}\frac{\partial a_3}{\partial t} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n,m+1}F_{n,m}$$

$$a_{24} = \left(\frac{\partial a_2}{\partial r} - \frac{s}{r}\frac{\partial a_2}{\partial s} - \frac{t}{r}\frac{\partial a_2}{\partial t} - \frac{u}{r}\frac{\partial a_2}{\partial u}\right) = \frac{1}{r}\frac{\partial a_4}{\partial t} = -\sum_{n=0}^{\infty'} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n+1,m+1}F_{n,m}$$

$$a_{33} = \frac{1}{r}\frac{\partial a_3}{\partial u} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n,m+2}D_{n,m}$$

$$a_{34} = \left(\frac{\partial a_3}{\partial r} - \frac{s}{r}\frac{\partial a_3}{\partial s} - \frac{t}{r}\frac{\partial a_3}{\partial t} - \frac{u}{r}\frac{\partial a_3}{\partial u}\right) = \frac{1}{r}\frac{\partial a_4}{\partial u} = -\sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n+1,m+2}D_{n,m}$$

$$a_{44} = \frac{\partial a_4}{\partial r} - \frac{s}{r}\frac{\partial a_4}{\partial s} - \frac{t}{r}\frac{\partial a_4}{\partial t} - \frac{u}{r}\frac{\partial a_4}{\partial u} - \frac{a_4}{r} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} [(n+2+m+1)A_{n+1,m+1} + uA_{n+1,m+2}]D_{n,m} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n+2,m+2}D_{n,m}$$

The nine summation coefficients, a_{ij}, together with the four summation coefficients given by Eqs. (30) and (30a) for the acceleration force vector, are the thirteen coefficients referred to in the summary of this report. Examination of Ref. 1, p. 6, shows that fourteen coefficients are required there.

The gradient of the vector F can be combined into a symmetric 3×3 matrix $P = \partial F/\partial R$, whose elements are given by

$$\begin{aligned} p_{11} &= a_{11} + s^2 a_{44} + a_4/r + 2sa_{14} \\ p_{12} &= p_{21} = a_{12} + sta_{44} + sa_{24} + ta_{14} \\ p_{13} &= p_{31} = a_{13} + sua_{44} + sa_{34} + ua_{14} \\ p_{22} &= -a_{11} + t^2 a_{44} + a_4/r + 2ta_{24} \\ p_{23} &= p_{32} = a_{23} + tua_{44} + ua_{24} + ta_{34} \\ p_{33} &= a_{33} + u^2 a_{44} + a_4/r + 2ua_{34} \end{aligned} \tag{37}$$

Before proceeding with the variational differential equations, we desire the gradient of the acceleration with respect to the mass coefficients, $C_{n,m}$ and $S_{n,m}$. We have

$$\begin{aligned} \frac{\partial F}{\partial C_{n,m}} &= \frac{\partial a_1}{\partial C_{n,m}}\hat{i} + \frac{\partial a_2}{\partial C_{n,m}}\hat{j} + \frac{\partial a_3}{\partial C_{n,m}}\hat{k} + \frac{\partial a_4}{\partial C_{n,m}}\hat{R} \\ \frac{\partial F}{\partial S_{n,m}} &= \frac{\partial a_1}{\partial S_{n,m}}\hat{i} + \frac{\partial a_2}{\partial S_{n,m}}\hat{j} + \frac{\partial a_3}{\partial S_{n,m}}\hat{k} + \frac{\partial a_4}{\partial S_{n,m}}\hat{R} \end{aligned} \tag{38}$$

The components of the $\partial F/\partial C_{n,m}$ and $\partial F/\partial S_{n,m}$ are given by

$$\begin{aligned} f_{C_{n,m,1}} &= (\rho_{n+2}/a)(mA_{n,m}r_{m-1} - sA_{n+1,m+1}r_m) \\ f_{C_{n,m,2}} &= -(\rho_{n+2}/a)(mA_{n,m}r_{m-1} + tA_{n+1,m+1}r_m) \\ f_{C_{n,m3}} &= (\rho_{n+2}/a)(A_{n,m+1} - uA_{n+1,m+1})r_m \\ f_{S_{n,m,1}} &= (\rho_{n+2}/a)(mA_{n,m}i_{m-1} - sA_{n+1,m+1}i_m) \\ f_{S_{n,m,2}} &= (\rho_{n+2}/a)(mA_{n,m}i_{m-1} - tA_{n+1,m+1}i_m) \\ f_{S_{n,m3}} &= (\rho_{n+2}/a)(A_{n,m+1} - uA_{n+1,m+1})i_m \end{aligned} \tag{39}$$

The Variational Differential Equations for a Rotating Body

Let the transformation between the Cartesian coordinate system fixed in the body be related to a true inertial Cartesian reference frame given by the rotation matrix $N(3\times3)$, where

$$R_B = NR_{IN} \tag{40}$$

The potential of a point mass in the coordinate system fixed in the body is defined as

$$\varphi = \varphi(R_B) \tag{41}$$

The acceleration force with respect to that body is defined as the gradient of that potential with respect to the coordinates of the point mass in the body-fixed system.

$$F_B = \partial\varphi(R_B)/\partial R_B \tag{42}$$

The acceleration force in the inertial system is then given by

$$F_{IN} = N^T F_B \tag{43}$$

The equations of motion of the point mass in the inertial system are given by

$$(d^2/dt^2)R_{IN} = N^T F_B \tag{44}$$

We require the variational differential equations for the partials of the inertial position and velocity components with respect to a generic variable γ_i such as the initial conditions and the mass coefficients. Differentiating Eq. (44), we obtain

$$\begin{aligned} (d/dt)(\partial R_{IN}/\partial\gamma_i) &= \partial\dot{R}_{IN}/\partial\gamma_i \\ (d/dt)(\partial\dot{R}_{IN}/\partial\gamma_i) &= N^T PN(\partial R_{IN}/\partial\gamma_i) + N^T(\partial F_B/\partial\gamma_i) \end{aligned} \tag{45}$$

The initial conditions for Eqs. (44) and (45) are

$$\begin{aligned} R_{IN}(t_0) &= R_0 \\ \dot{R}_{IN}(t_0) &= \dot{R}_0 \\ \frac{\partial R_{IN}(t_0)}{\partial\gamma_i} &= \begin{matrix} 1 & \text{if} & \gamma_i = r_i(0) \\ 0 & \text{if} & \gamma_i \neq r_i(0) \end{matrix} \\ \frac{\partial\dot{R}_{IN}(t_0)}{\partial\gamma_i} &= \begin{matrix} 1 & \text{if} & \gamma_i = v_i(0) \\ 0 & \text{if} & \gamma_i \neq v_i(0) \end{matrix} \end{aligned} \tag{46}$$

References

[1] Gulick, L. J., "A Comparison of Methods for Computing Gravitational Potential Derivatives," TR C & GS 40, Sept. 1970, U.S. Dept. of Commerce Environmental Science Services Administration, Washington, D.C.

[2] Vinti, J. P., "Improvement of the Spheroidal Method for Artificial Satellites," *Astronomical Journal*, Vol. 74, No. 1, Feb. 1969, pp. 25–34.